

The Cobordism Hypothesis in Dimension 1

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Abstract

In [Lur1] Lurie published an expository article outlining a proof for a higher version of the cobordism hypothesis conjectured by Baez and Dolan in [BaDo]. In this note we give a proof for the 1-dimensional case of this conjecture. The proof follows most of the outline given in [Lur1], but differs in a few crucial details. In particular, the proof makes use of the theory of quasi-unital ∞ -categories as developed by the author in [Har].

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1 Introduction

Let $\mathcal{B}_1^{\text{or}}$ denote the 1-dimensional oriented cobordism ∞ -category, i.e. the symmetric monoidal ∞ -category whose objects are oriented 0-dimensional closed manifolds and whose morphisms are oriented 1-dimensional cobordisms between them.

Let \mathcal{D} be a symmetric monoidal ∞ -category with duals. The 1-dimensional cobordism hypothesis concerns the ∞ -category

$$\text{Fun}^{\otimes}(\mathcal{B}_1^{\text{or}}, \mathcal{D})$$

of symmetric monoidal functors $\varphi : \mathcal{B}_1^{\text{or}} \rightarrow \mathcal{D}$. If $X_+ \in \mathcal{B}_1^{\text{or}}$ is the object corresponding to a point with positive orientation then the evaluation map $Z \mapsto Z(X_+)$ induces a functor

$$\text{Fun}^{\otimes}(\mathcal{B}_1^{\text{or}}, \mathcal{D}) \rightarrow \mathcal{D}$$

It is not hard to show that since $\mathcal{B}_1^{\text{or}}$ has duals the ∞ -category $\text{Fun}^{\otimes}(\mathcal{B}_1^{\text{or}}, \mathcal{D})$ is in fact an ∞ -groupoid, i.e. every natural transformation between two functors

$F, G : \mathcal{B}_1^{\text{or}} \longrightarrow \mathcal{D}$ is a natural equivalence. This means that the evaluation map $Z \mapsto Z(X_+)$ actually factors through a map

$$\text{Fun}^{\otimes}(\mathcal{B}_1^{\text{or}}, \mathcal{D}) \longrightarrow \widetilde{\mathcal{D}}$$

where $\widetilde{\mathcal{D}}$ is the maximal ∞ -groupoid of \mathcal{D} . The cobordism hypothesis then states

Theorem 1.1. *The evaluation map*

$$\text{Fun}^{\otimes}(\mathcal{B}_1^{\text{or}}, \mathcal{D}) \longrightarrow \widetilde{\mathcal{D}}$$

is an equivalence of ∞ -categories.

Remark 1.2. From the consideration above we see that we could have written the cobordism hypothesis as an equivalence

$$\widetilde{\text{Fun}}^{\otimes}(\mathcal{B}_1^{\text{or}}, \mathcal{D}) \xrightarrow{\simeq} \widetilde{\mathcal{D}}$$

where $\widetilde{\text{Fun}}^{\otimes}(\mathcal{B}_1^{\text{or}}, \mathcal{D})$ is the maximal ∞ -groupoid of $\text{Fun}^{\otimes}(\mathcal{B}_1^{\text{or}}, \mathcal{D})$ (which in this case happens to coincide with $\text{Fun}^{\otimes}(\mathcal{B}_1^{\text{or}}, \mathcal{D})$). This ∞ -groupoid is the fundamental groupoid of the space of maps from $\mathcal{B}_1^{\text{or}}$ to \mathcal{D} in the ∞ -category Cat^{\otimes} of symmetric monoidal ∞ -categories.

In his paper [Lur1] Lurie gives an elaborate sketch of proof for a higher dimensional generalization of the 1-dimensional cobordism hypothesis. For this one needs to generalize the notion of ∞ -categories to (∞, n) -categories. The strategy of proof described in [Lur1] is inductive in nature. In particular in order to understand the $n = 1$ case, one should start by considering the $n = 0$ case.

Let $\mathcal{B}_0^{\text{un}}$ be the 0-dimensional unoriented cobordism category, i.e. the objects of $\mathcal{B}_0^{\text{un}}$ are 0-dimensional closed manifolds (or equivalently, finite sets) and the morphisms are diffeomorphisms (or equivalently, isomorphisms of finite sets). Note that $\mathcal{B}_0^{\text{un}}$ is a (discrete) ∞ -groupoid.

Let $X \in \mathcal{B}_0^{\text{un}}$ be the object corresponding to one point. Then the 0-dimensional cobordism hypothesis states that $\mathcal{B}_0^{\text{un}}$ is in fact the free ∞ -groupoid (or $(\infty, 0)$ -category) on one object, i.e. if \mathcal{G} is any other ∞ -groupoid then the evaluation map $Z \mapsto Z(X)$ induces an equivalence of ∞ -groupoids

$$\text{Fun}^{\otimes}(\mathcal{B}_0^{\text{un}}, \mathcal{G}) \xrightarrow{\simeq} \mathcal{G}$$

Remark 1.3. At this point one can wonder what is the justification for considering non-oriented manifolds in the $n = 0$ case oriented ones in the $n = 1$ case. As is explained in [Lur1] the desired notion when working in the n -dimensional cobordism (∞, n) -category is that of **n -framed** manifolds. One then observes that 0-framed 0-manifolds are unoriented manifolds, while taking 1-framed 1-manifolds (and 1-framed 0-manifolds) is equivalent to taking the respective manifolds with orientation.

Now the 0-dimensional cobordism hypothesis is not hard to verify. In fact, it holds in a slightly more general context - we do not have to assume that \mathcal{G} is an ∞ -groupoid. In fact, if \mathcal{G} is **any symmetric monoidal ∞ -category** then the evaluation map induces an equivalence of ∞ -categories

$$\mathrm{Fun}^{\otimes}(\mathcal{B}_0^{\mathrm{un}}, \mathcal{G}) \xrightarrow{\simeq} \mathcal{G}$$

and hence also an equivalence of ∞ -groupoids

$$\widetilde{\mathrm{Fun}}^{\otimes}(\mathcal{B}_0^{\mathrm{un}}, \mathcal{G}) \xrightarrow{\simeq} \widetilde{\mathcal{G}}$$

Now consider the under-category $\mathrm{Cat}_{\mathcal{B}_0^{\mathrm{un}}/}^{\otimes}$ of symmetric monoidal ∞ -categories \mathcal{D} equipped with a functor $\mathcal{B}_0^{\mathrm{un}} \rightarrow \mathcal{D}$. Since $\mathcal{B}_0^{\mathrm{un}}$ is free on one generator this category can be identified with the ∞ -category of **pointed** symmetric monoidal ∞ -categories, i.e. symmetric monoidal ∞ -categories with a chosen object. We will often not distinguish between these two notions.

Now the point of positive orientation $X_+ \in \mathcal{B}_1^{\mathrm{or}}$ determines a functor $\mathcal{B}_0^{\mathrm{un}} \rightarrow \mathcal{B}_1^{\mathrm{or}}$, i.e. an object in $\mathrm{Cat}_{\mathcal{B}_0^{\mathrm{un}}/}^{\otimes}$, which we shall denote by \mathcal{B}_1^+ . The 1-dimensional cobordism hypothesis is then equivalent to the following statement:

Theorem 1.4. *[Cobordism Hypothesis 0-to-1] Let $\mathcal{D} \in \mathrm{Cat}_{\mathcal{B}_0^{\mathrm{un}}/}^{\otimes}$ be a pointed symmetric monoidal ∞ -category with duals. Then the ∞ -groupoid*

$$\widetilde{\mathrm{Fun}}_{\mathcal{B}_0^{\mathrm{un}}/}^{\otimes}(\mathcal{B}_1^+, \mathcal{D})$$

is contractible.

Theorem 1.4 can be considered as the inductive step from the 0-dimensional cobordism hypothesis to the 1-dimensional one. Now the strategy outlined in [Lur1] proceeds to bridge the gap between $\mathcal{B}_0^{\mathrm{un}}$ to $\mathcal{B}_1^{\mathrm{or}}$ by considering an intermediate ∞ -category

$$\mathcal{B}_0^{\mathrm{un}} \hookrightarrow \mathcal{B}_1^{\mathrm{ev}} \hookrightarrow \mathcal{B}_1^{\mathrm{or}}$$

This intermediate ∞ -category is defined in [Lur1] in terms of framed functions and index restriction. However in the 1-dimensional case one can describe it without going into the theory of framed functors. In particular we will use the following definition:

Definition 1.5. Let $\iota : \mathcal{B}_1^{\mathrm{ev}} \hookrightarrow \mathcal{B}_1^{\mathrm{or}}$ be the subcategory containing all objects and only the cobordisms M in which every connected component $M_0 \subseteq M$ is either an identity segment or an evaluation segment.

Let us now describe how to bridge the gap between $\mathcal{B}_0^{\mathrm{un}}$ and $\mathcal{B}_1^{\mathrm{ev}}$. Let \mathcal{D} be an ∞ -category with duals and let

$$\varphi : \mathcal{B}_1^{\mathrm{ev}} \rightarrow \mathcal{D}$$

be a symmetric monoidal functor. We will say that φ is **non-degenerate** if for each $X \in \mathcal{B}_1^{\mathrm{ev}}$ the map

$$\varphi(\mathrm{ev}_X) : \varphi(X) \otimes \varphi(\check{X}) \simeq \varphi(X \otimes \check{X}) \rightarrow \varphi(1) \simeq 1$$

is **non-degenerate**, i.e. identifies $\varphi(\check{X})$ with a dual of $\varphi(X)$. We will denote by

$$\mathrm{Cat}_{\mathcal{B}_1^{\mathrm{ev}}/}^{\mathrm{nd}} \subseteq \mathrm{Cat}_{\mathcal{B}_1^{\mathrm{ev}}/}^{\otimes}$$

the full subcategory spanned by objects $\varphi : \mathcal{B}_1^{\mathrm{ev}} \rightarrow \mathcal{D}$ such that \mathcal{D} has duals and φ is non-degenerate.

Let $X_+ \in \mathcal{B}_1^{\mathrm{ev}}$ be the point with positive orientation. Then X_+ determines a functor

$$\mathcal{B}_0^{\mathrm{un}} \rightarrow \mathcal{B}_1^{\mathrm{ev}}$$

The restriction map $\varphi \mapsto \varphi|_{\mathcal{B}_0^{\mathrm{un}}}$ then induces a functor

$$\mathrm{Cat}_{\mathcal{B}_1^{\mathrm{ev}}/}^{\mathrm{nd}} \rightarrow \mathrm{Cat}_{\mathcal{B}_0^{\mathrm{un}}/}^{\otimes}$$

Now the gap between $\mathcal{B}_1^{\mathrm{ev}}$ and $\mathcal{B}_0^{\mathrm{un}}$ can be climbed using the following lemma (see [Lur1]):

Lemma 1.6. *The functor*

$$\mathrm{Cat}_{\mathcal{B}_1^{\mathrm{ev}}/}^{\mathrm{nd}} \rightarrow \mathrm{Cat}_{\mathcal{B}_0^{\mathrm{un}}/}^{\otimes}$$

is fully faithful.

Proof. First note that if $F : \mathcal{D} \rightarrow \mathcal{D}'$ is a symmetric monoidal functor where $\mathcal{D}, \mathcal{D}'$ have duals and $\varphi : \mathcal{B}_1^{\mathrm{ev}} \rightarrow \mathcal{D}$ is non-degenerate then $f \circ \varphi$ will be non-degenerate as well. Hence it will be enough to show that if \mathcal{D} has duals then the restriction map induces an equivalence between the ∞ -groupoid of non-degenerate symmetric monoidal functors

$$\mathcal{B}_1^{\mathrm{ev}} \rightarrow \mathcal{D}$$

and the ∞ -groupoid of symmetric monoidal functors

$$\mathcal{B}_0^{\mathrm{un}} \rightarrow \mathcal{D}$$

Now specifying a non-degenerate functor

$$\mathcal{B}_1^{\mathrm{ev}} \rightarrow \mathcal{D}$$

is equivalent to specifying a pair of objects $D_+, D_- \in \mathcal{D}$ (the images of X_+, X_- respectively) and a non-degenerate morphism

$$e : D_+ \otimes D_- \rightarrow 1$$

which is the image of ev_{X_+} . Since \mathcal{D} has duals the ∞ -groupoid of triples (D_+, D_-, e) in which e is non-degenerate is equivalent to the ∞ -groupoid of triples (D_+, \check{D}_-, f) where $f : D_+ \rightarrow \check{D}_-$ is an equivalence. Hence the forgetful map $(D_+, D_-, e) \mapsto D_+$ is an equivalence. \square

Now consider the natural inclusion $\iota : \mathcal{B}_1^{\text{ev}} \longrightarrow \mathcal{B}_1^{\text{or}}$ as an object in $\text{Cat}_{\mathcal{B}_1^{\text{ev}}/}^{\text{nd}}$. Then by Lemma 1.6 we see that the 1-dimensional cobordism hypothesis will be established once we make the following last step:

Theorem 1.7 (Cobordism Hypothesis - Last Step). *Let \mathcal{D} be a symmetric monoidal ∞ -category with duals and let $\varphi : \mathcal{B}_1^{\text{ev}} \longrightarrow \mathcal{D}$ be a **non-degenerate** functor. Then the ∞ -groupoid*

$$\widetilde{\text{Fun}}_{\mathcal{B}_1^{\text{ev}}/}^{\otimes}(\mathcal{B}_1^{\text{or}}, \mathcal{D})$$

is contractible.

Note that since $\mathcal{B}_1^{\text{ev}} \longrightarrow \mathcal{B}_1^{\text{or}}$ is essentially surjective all the functors in

$$\widetilde{\text{Fun}}_{\mathcal{B}_1^{\text{ev}}/}^{\otimes}(\mathcal{B}_1^{\text{or}}, \mathcal{D})$$

will have the same essential image of φ . Hence it will be enough to prove for the claim for the case where $\varphi : \mathcal{B}_1^{\text{ev}} \longrightarrow \mathcal{D}$ is **essentially surjective**. We will denote by

$$\text{Cat}_{\mathcal{B}_1^{\text{ev}}/}^{\text{sur}} \subseteq \text{Cat}_{\mathcal{B}_1^{\text{ev}}/}^{\text{nd}}$$

the full subcategory spanned by essentially surjective functors $\varphi : \mathcal{B}_1^{\text{ev}} \longrightarrow \mathcal{D}$. Hence we can phrase Theorem 1.7 as follows:

Theorem 1.8 (Cobordism Hypothesis - Last Step 2). *Let \mathcal{D} be a symmetric monoidal ∞ -category with duals and let $\varphi : \mathcal{B}_1^{\text{ev}} \longrightarrow \mathcal{D}$ be an **essentially surjective non-degenerate** functor. Then the space of maps*

$$\text{Map}_{\text{Cat}_{\mathcal{B}_1^{\text{ev}}/}^{\text{sur}}}(\iota, \varphi)$$

is contractible.

The purpose of this paper is to provide a formal proof for this last step. This paper is constructed as follows. In § 2 we prove a variant of Theorem 1.8 which we call the quasi-unital cobordism hypothesis (Theorem 2.6). Then in § 3 we explain how to deduce Theorem 1.8 from Theorem 2.6. Section § 3 relies on the notion of **quasi-unital ∞ -categories** which is developed rigourously in [Har] (however § 2 is completely independent of [Har]).

2 The Quasi-Unital Cobordism Hypothesis

Let $\varphi : \mathcal{B}_1^{\text{ev}} \longrightarrow \mathcal{D}$ be a non-degenerate functor and let Grp_{∞} denote the ∞ -category of ∞ -groupoids. We can define a lax symmetric functor $M_{\varphi} : \mathcal{B}_1^{\text{ev}} \longrightarrow \text{Grp}_{\infty}$ by setting

$$M_{\varphi}(X) = \text{Map}_{\mathcal{D}}(1, \varphi(X))$$

We will refer to M_{φ} as the **fiber functor** of φ . Now if \mathcal{D} has duals and φ is non-degenerate, then one can expect this to be reflected in M_{φ} somehow. More precisely, we have the following notion:

Definition 2.1. Let $M : \mathcal{B}_1^{\text{ev}} \rightarrow \text{Grp}_\infty$ be a lax symmetric monoidal functor. An object $Z \in M(X \otimes \check{X})$ is called **non-degenerate** if for each object $Y \in \mathcal{B}_1^{\text{ev}}$ the natural map

$$M(Y \otimes \check{X}) \xrightarrow{Id \times Z} M(Y \otimes \check{X}) \times M(X \otimes \check{X}) \rightarrow M(Y \otimes \check{X} \otimes X \otimes \check{X}) \xrightarrow{M(Id \otimes \text{ev} \otimes Id)} M(Y \otimes \check{X})$$

is an equivalence of ∞ -groupoids.

Remark 2.2. If a non-degenerate element $Z \in M(X \otimes \check{X})$ exists then it is unique up to a (non-canonical) equivalence.

Example 1. Let $M : \mathcal{B}_1^{\text{ev}} \rightarrow \text{Grp}_\infty$ be a lax symmetric monoidal functor. The lax symmetric structure of M includes a structure map $1_{\text{Grp}_\infty} \rightarrow M(1)$ which can be described by choosing an object $Z_1 \in M(1)$. The axioms of lax monoidality then ensure that Z_1 is non-degenerate.

Definition 2.3. A lax symmetric monoidal functor $M : \mathcal{B}_1^{\text{ev}} \rightarrow \text{Grp}_\infty$ will be called **non-degenerate** if for each object $X \in \mathcal{B}_1^{\text{ev}}$ there exists a non-degenerate object $Z \in M(X \otimes \check{X})$.

Definition 2.4. Let $M_1, M_2 : \mathcal{B}_1^{\text{ev}} \rightarrow \text{Grp}_\infty$ be two non-degenerate lax symmetric monoidal functors. A lax symmetric natural transformation $T : M_1 \rightarrow M_2$ will be called **non-degenerate** if for each object $X \in \text{Bord}^{\text{ev}}$ and each non-degenerate object $Z \in M_1(X \otimes \check{X})$ the objects $T(Z) \in M_2(X \otimes \check{X})$ is non-degenerate.

Remark 2.5. From remark 2.2 we see that if $T(Z) \in M_2(X \otimes \check{X})$ is non-degenerate for **at least one** non-degenerate $Z \in M_1(X \otimes \check{X})$ then it will be true for all non-degenerate $Z \in M_1(X \otimes \check{X})$.

Now we claim that if \mathcal{D} has duals and $\varphi : \mathcal{B}_1^{\text{ev}} \rightarrow \mathcal{D}$ is non-degenerate then the fiber functor M_φ will be non-degenerate: for each object $X \in \mathcal{B}_1^{\text{ev}}$ there exists a coevaluation morphism

$$\text{coev}_{\varphi(X)} : 1 \rightarrow \varphi(X) \otimes \varphi(\check{X}) \simeq \varphi(X \otimes \check{X})$$

which determines an element in $Z_X \in M_\varphi(X \otimes \check{X})$. It is not hard to see that this element is non-degenerate.

Let $\text{Fun}^{\text{lax}}(\mathcal{B}_1^{\text{ev}}, \text{Grp}_\infty)$ denote the ∞ -category of lax symmetric monoidal functors $\mathcal{B}_1^{\text{ev}} \rightarrow \text{Grp}_\infty$ and by

$$\text{Fun}_{\text{nd}}^{\text{lax}}(\mathcal{B}_1^{\text{ev}}, \text{Grp}_\infty) \subseteq \text{Fun}^{\text{lax}}(\mathcal{B}_1^{\text{ev}}, \text{Grp}_\infty)$$

the subcategory spanned by non-degenerate functors and non-degenerate natural transformations. Now the construction $\varphi \mapsto M_\varphi$ determines a functor

$$\text{Cat}_{\mathcal{B}_1^{\text{ev}}}^{\text{nd}} \rightarrow \text{Fun}_{\text{nd}}^{\text{lax}}(\mathcal{B}_1^{\text{ev}}, \text{Grp}_\infty)$$

In particular if $\varphi : \mathcal{B}_1^{\text{ev}} \rightarrow \mathcal{C}$ and $\psi : \mathcal{B}_1^{\text{ev}} \rightarrow \mathcal{D}$ are non-degenerate then any functor $T : \mathcal{C} \rightarrow \mathcal{D}$ under $\mathcal{B}_1^{\text{ev}}$ will induce a non-degenerate natural transformation

$$T_* : M_\varphi \rightarrow M_\psi$$

The rest of this section is devoted to proving the following result, which we call the "quasi-unital cobordism hypothesis":

Theorem 2.6 (Cobordism Hypothesis - Quasi-Unital). *Let \mathcal{D} be a symmetric monoidal ∞ -category with duals, let $\varphi : \mathcal{B}_1^{\text{ev}} \rightarrow \mathcal{D}$ be a non-degenerate functor and let $\iota : \mathcal{B}_1^{\text{ev}} \hookrightarrow \mathcal{B}_1^{\text{or}}$ be the natural inclusion. Let $M_\iota, M_\varphi \in \text{Fun}_{\text{nd}}^{\text{lax}}$ be the corresponding fiber functors. Then the space of maps*

$$\text{Map}_{\text{Fun}_{\text{nd}}^{\text{lax}}}(M_\iota, M_\varphi)$$

is contractible.

Proof. We start by transforming the lax symmetric monoidal functors M_ι, M_φ to **left fibrations** over $\mathcal{B}_1^{\text{ev}}$ using the symmetric monoidal analogue of Grothendieck's construction, as described in [Lur1], page 67 – 68.

Let $M : \mathcal{B} \rightarrow \text{Grp}_\infty$ be a lax symmetric monoidal functor. We can construct a symmetric monoidal ∞ -category $\text{Groth}(\mathcal{B}, M)$ as follows:

1. The objects of $\text{Groth}(\mathcal{B}, M)$ are pairs (X, η) where $X \in \mathcal{B}$ is an object and η is an object of $M(X)$.
2. The space of maps from (X, η) to (X', η') in $\text{Groth}(\mathcal{B}, M)$ is defined to be the classifying space of the ∞ -groupoid of pairs (f, α) where $f : X \rightarrow X'$ is a morphism in \mathcal{B} and $\alpha : f_*\eta \rightarrow \eta'$ is a morphism in $M(X')$. Composition is defined in a straightforward way.
3. The symmetric monoidal structure on $\text{Groth}(\mathcal{B}, M)$ is obtained by defining

$$(X, \eta) \otimes (X', \eta') = (X \otimes X', \beta_{X, Y}(\eta \otimes \eta'))$$

where $\beta_{X, Y} : M(X) \times M(Y) \rightarrow M(X \otimes Y)$ is given by the lax symmetric structure of M .

The forgetful functor $(X, \eta) \mapsto X$ induces a **left fibration**

$$\text{Groth}(\mathcal{B}, M) \rightarrow \mathcal{B}$$

Theorem 2.7. *The association $M \mapsto \text{Groth}(\mathcal{B}, M)$ induces an equivalence between the ∞ -category of lax-symmetric monoidal functors $\mathcal{B} \rightarrow \text{Grp}_\infty$ and the full subcategory of the over ∞ -category $\text{Cat}_{/\mathcal{B}}^\otimes$ spanned by left fibrations.*

Proof. This follows from the more general statement given in [Lur1] Proposition 3.3.26. Note that any map of left fibrations over \mathcal{B} is in particular a map of coCartesian fibrations because if $p : \mathcal{C} \rightarrow \mathcal{B}$ is a left fibration then any edge in \mathcal{C} is p -coCartesian. \square

Remark 2.8. Note that if $\mathcal{C} \rightarrow \mathcal{B}$ is a left fibration of symmetric monoidal ∞ -categories and $\mathcal{A} \rightarrow \mathcal{B}$ is a symmetric monoidal functor then the ∞ -category

$$\text{Fun}_{/\mathcal{B}}^\otimes(\mathcal{A}, \mathcal{C})$$

is actually an ∞ -**groupoid**, and by Theorem 2.7 is equivalent to the ∞ -groupoid of lax-monoidal natural transformations between the corresponding lax monoidal functors from \mathcal{B} to \mathbf{Grp}_∞ .

Now set

$$\mathcal{F}_\iota \stackrel{\text{def}}{=} \text{Groth}(\mathcal{B}_1^{\text{ev}}, M_\iota)$$

$$\mathcal{F}_\varphi \stackrel{\text{def}}{=} \text{Groth}(\mathcal{B}_1^{\text{ev}}, M_\varphi)$$

Let

$$\text{Fun}_{/\mathcal{B}_1^{\text{ev}}}^{\text{nd}}(\mathcal{F}_\iota, \mathcal{F}_\varphi) \subseteq \text{Fun}_{/\mathcal{B}_1^{\text{ev}}}^\otimes(\mathcal{F}_\iota, \mathcal{F}_\varphi)$$

denote the full sub ∞ -groupoid of functors which correspond to **non-degenerate** natural transformations

$$M_\iota \longrightarrow M_\varphi$$

under the Grothendieck construction. Note that $\text{Fun}_{/\mathcal{B}_1^{\text{ev}}}^{\text{nd}}(\mathcal{F}_\iota, \mathcal{F}_\varphi)$ is a union of connected components of the ∞ -groupoid $\text{Fun}_{/\mathcal{B}_1^{\text{ev}}}^\otimes(\mathcal{F}_\iota, \mathcal{F}_\varphi)$.

We now need to show that the ∞ -groupoid

$$\text{Fun}_{/\mathcal{B}_1^{\text{ev}}}^{\text{nd}}(\mathcal{F}_\iota, \mathcal{F}_\varphi)$$

is contractible.

Unwinding the definitions we see that the objects of \mathcal{F}_ι are pairs (X, M) where $X \in \mathcal{B}_1^{\text{ev}}$ is a 0-manifold and $M \in \text{Map}_{\mathcal{B}_1^{\text{gr}}}(\emptyset, X)$ is a cobordism from \emptyset to X . A morphism in φ from (X, M) to (X', M') consists of a morphism in $\mathcal{B}_1^{\text{ev}}$

$$N : X \longrightarrow X'$$

and a diffeomorphism

$$T : M \coprod_X N \cong M'$$

respecting X' . Note that for each $(X, M) \in \mathcal{F}_\iota$ we have an identification $X \simeq \partial M$. Further more the space of morphisms from $(\partial M, M)$ to $(\partial M', M')$ is **homotopy equivalent to the space of orientation-preserving π_0 -surjective embeddings of M in M'** (which are not required to respect the boundaries in any way).

Now in order to analyze the symmetric monoidal ∞ -category \mathcal{F}_ι we are going to use the theory of ∞ -**operads**, as developed in [Lur2]. Recall that the category Cat^\otimes of symmetric monoidal ∞ -categories admits a forgetful functor

$$\text{Cat}^\otimes \longrightarrow \text{Op}^\infty$$

to the ∞ -category of ∞ -**operads**. This functor has a left adjoint

$$\text{Env} : \text{Op}^\infty \longrightarrow \text{Cat}^\otimes$$

called the **monoidal envelope** functor (see [Lur2] §2.2.4). In particular, if \mathcal{C}^\otimes is an ∞ -operad and \mathcal{D} is a symmetric monoidal ∞ -category with corresponding ∞ -operad $\mathcal{D}^\otimes \longrightarrow \mathbf{N}(\Gamma_*)$ then there is an **equivalence of ∞ -categories**

$$\text{Fun}^\otimes(\text{Env}(\mathcal{C}^\otimes), \mathcal{D}) \simeq \text{Alg}_{\mathcal{C}}(\mathcal{D}^\otimes)$$

Where $\text{Alg}_{\mathcal{C}}(\mathcal{D}^{\otimes}) \subseteq \text{Fun}_{/\text{N}(\Gamma_*)}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ denotes the full subcategory spanned by ∞ -operad maps (see Proposition 2.2.4.9 of [Lur2]).

Now observing the definition of monoidal envelop (see Remark 2.2.4.3 in [Lur2]) we see that \mathcal{F}_l is equivalent to the monoidal envelope of a certain simple ∞ -operad

$$F_l \simeq \text{Env}(\mathcal{OF}^{\otimes})$$

which can be described as follows: the underlying ∞ -category \mathcal{OF} of \mathcal{OF}^{\otimes} is the ∞ -category of **connected** 1-manifolds (i.e. either the segment or the circle) and the morphisms are **orientation-preserving embeddings** between them. The (active) n -to-1 operations of \mathcal{OF} (for $n \geq 1$) from (M_1, \dots, M_n) to M are the orientation-preserving embeddings

$$M_1 \amalg \dots \amalg M_n \longrightarrow M$$

and there are no 0-to-1 operations.

Now observe that the induced map $\mathcal{OF}^{\otimes} \longrightarrow (\mathcal{B}_1^{\text{ev}})^{\infty}$ is a fibration of ∞ -operads. We claim that \mathcal{F}_l is not only the enveloping symmetric monoidal ∞ -category of \mathcal{OF}^{\otimes} , but that $\mathcal{F}_l \longrightarrow \mathcal{B}_1^{\text{ev}}$ is the enveloping **left fibration** of $\mathcal{OF} \longrightarrow \mathcal{B}_1^{\text{ev}}$. More precisely we claim that for any left fibration $\mathcal{D} \longrightarrow \mathcal{B}_1^{\text{ev}}$ of symmetric monoidal ∞ -categories the natural map

$$\text{Fun}_{/\mathcal{B}_1^{\text{ev}}}^{\otimes}(F_l, \mathcal{D}) \longrightarrow \text{Alg}_{\mathcal{OF}/\mathcal{B}_1^{\text{ev}}}(\mathcal{D}^{\otimes})$$

is an equivalence if ∞ -groupoids (where both terms denote mapping objects in the respective **over-categories**). This is in fact not a special property of F_l :

Lemma 2.9. *Let \mathcal{O} be a symmetric monoidal ∞ -category with corresponding ∞ -operad $\mathcal{O}^{\otimes} \longrightarrow \text{N}(\Gamma_*)$ and let $p : \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ be a fibration of ∞ -operads such that the induced map*

$$\bar{p} : \text{Env}(\mathcal{C}^{\otimes}) \longrightarrow \mathcal{O}$$

is a left fibration. Let $\mathcal{D} \longrightarrow \mathcal{O}$ be some other left fibration of symmetric monoidal categories. Then the natural map

$$\text{Fun}_{/\mathcal{O}}^{\otimes}(\text{Env}(\mathcal{C}^{\otimes}), \mathcal{D}) \longrightarrow \text{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D}^{\otimes})$$

is an equivalence of ∞ -categories. Further more both sides are in fact ∞ -groupoids.

Proof. Consider the diagram

$$\begin{array}{ccc} \text{Fun}^{\otimes}(\text{Env}(\mathcal{C}^{\otimes}), \mathcal{D}) & \xrightarrow{\simeq} & \text{Alg}_{\mathcal{C}}(\mathcal{D}^{\otimes}) \\ \downarrow & & \downarrow \\ \text{Fun}^{\otimes}(\text{Env}(\mathcal{C}^{\otimes}), \mathcal{O}) & \xrightarrow{\simeq} & \text{Alg}_{\mathcal{C}}(\mathcal{O}^{\otimes}) \end{array}$$

Now the vertical maps are left fibrations and by adjunction the horizontal maps are equivalences. By [Lur3] Proposition 3.3.1.5 we get that the induced map on the fibers of p and \bar{p} respectively

$$\mathrm{Fun}_{/\mathcal{O}}^{\otimes}(\mathrm{Env}(\mathcal{C}^{\otimes}), \mathcal{D}) \longrightarrow \mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D}^{\otimes})$$

is a weak equivalence of ∞ -groupoids. \square

Remark 2.10. In [Lur2] a relative variant $\mathrm{Env}_{\mathcal{B}_1^{\mathrm{ev}}}$ of Env is introduced which sends a fibration of ∞ -operads $\mathcal{C}^{\otimes} \rightarrow (\mathcal{B}_1^{\mathrm{ev}})^{\otimes}$ to its enveloping coCartesian fibration $\mathrm{Env}_{\mathcal{O}}(\mathcal{C}^{\otimes}) \rightarrow \mathcal{B}_1^{\mathrm{ev}}$. Note that in our case the map

$$\mathcal{F}_\iota \rightarrow \mathcal{B}_1^{\mathrm{ev}}$$

is **not** the enveloping coCartesian fibration of $\mathcal{O}\mathcal{F}^{\otimes} \rightarrow (\mathcal{B}_1^{\mathrm{ev}})^{\otimes}$. However from Lemma 2.9 it follows that the map

$$\begin{array}{ccc} \mathcal{F}_\iota & \xrightarrow{\quad} & \mathrm{Env}_{\mathcal{B}_1^{\mathrm{ev}}}(\mathcal{O}\mathcal{F}^{\otimes}) \\ & \searrow & \swarrow \\ & \mathcal{B}_1^{\mathrm{ev}} & \end{array}$$

is a **covariant equivalence** over $\mathcal{B}_1^{\mathrm{ev}}$, i.e. induces a weak equivalence of simplicial sets on the fibers (where the fibers on the left are ∞ -groupoids and the fibers on the right are ∞ -categories). This claim can also be verified directly by unwinding the definition of $\mathrm{Env}_{\mathcal{B}_1^{\mathrm{ev}}}(\mathcal{O}\mathcal{F}^{\otimes})$.

Summing up the discussion so far we observe that we have a weak equivalence of ∞ -groupoids

$$\mathrm{Fun}_{/\mathcal{B}_1^{\mathrm{ev}}}^{\otimes}(\mathcal{F}_\iota, \mathcal{F}_\varphi) \xrightarrow{\simeq} \mathrm{Alg}_{\mathcal{O}\mathcal{F}/\mathcal{B}_1^{\mathrm{ev}}}(\mathcal{F}_\varphi^{\otimes})$$

Let

$$\mathrm{Alg}_{\mathcal{O}\mathcal{F}/\mathcal{B}_1^{\mathrm{ev}}}^{\mathrm{nd}}(\mathcal{F}_\varphi^{\otimes}) \subseteq \mathrm{Alg}_{\mathcal{O}\mathcal{F}/\mathcal{B}_1^{\mathrm{ev}}}(\mathcal{F}_\varphi^{\otimes})$$

denote the full sub ∞ -groupoid corresponding to

$$\mathrm{Fun}_{/\mathcal{B}_1^{\mathrm{ev}}}^{\mathrm{nd}}(\mathcal{F}_\iota, \mathcal{F}_\varphi) \subseteq \mathrm{Fun}_{/\mathcal{B}_1^{\mathrm{ev}}}^{\otimes}(\mathcal{F}_\iota, \mathcal{F}_\varphi)$$

under the adjunction. We are now reduced to prove that the ∞ -groupoid

$$\mathrm{Alg}_{\mathcal{O}\mathcal{F}/\mathcal{B}_1^{\mathrm{ev}}}^{\mathrm{nd}}(\mathcal{F}_\varphi^{\otimes})$$

is contractible.

Let $\mathcal{O}\mathcal{J}^{\otimes} \subseteq \mathcal{O}\mathcal{F}^{\otimes}$ be the full sub ∞ -operad of $\mathcal{O}\mathcal{F}^{\otimes}$ spanned by connected 1-manifolds which are diffeomorphic to the segment (and all n -to-1 operations between them). In particular we see that $\mathcal{O}\mathcal{J}^{\otimes}$ is equivalent to the **non-unital associative ∞ -operad**.

We begin with the following theorem which reduces the handling of $\mathcal{O}\mathcal{F}^{\otimes}$ to $\mathcal{O}\mathcal{J}^{\otimes}$.

Theorem 2.11. *Let $q : \mathcal{C}^\otimes \longrightarrow \mathcal{O}^\otimes$ be a left fibration of ∞ -operads. Then the restriction map*

$$\mathrm{Alg}_{\mathcal{O}\mathcal{F}/\mathcal{O}}(\mathcal{C}^\otimes) \longrightarrow \mathrm{Alg}_{\mathcal{O}\mathcal{J}/\mathcal{O}}(\mathcal{C}^\otimes)$$

is a weak equivalence.

Proof. We will base our claim on the following general lemma:

Lemma 2.12. *Let $\mathcal{A}^\otimes \longrightarrow \mathcal{B}^\otimes$ be a map of ∞ -groupoids and let $q : \mathcal{C}^\otimes \longrightarrow \mathcal{O}^\otimes$ be **left fibration** of ∞ -operads. Suppose that for every object $B \in \mathcal{B}$, the category*

$$\mathcal{F}_B = \mathcal{A}_{\mathrm{act}}^\otimes \times_{\mathcal{B}_{\mathrm{act}}^\otimes} \mathcal{B}_{/B}^\otimes$$

is weakly contractible (see [Lur2] for the terminology). Then the natural restriction map

$$\mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathcal{C}^\otimes) \longrightarrow \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C}^\otimes)$$

is a weak equivalence.

Proof. In [Lur2] §3.1.3 it is explained how under certain conditions the forgetful functor (i.e. restriction map)

$$\mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathcal{C}^\otimes) \longrightarrow \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C}^\otimes)$$

admits a left adjoint, called the **free algebra functor**. Since $\mathcal{C}^\otimes \longrightarrow \mathcal{O}^\otimes$ is a left fibration both these ∞ -categories are ∞ -groupoids, and so any adjunction between them will be an equivalence. Hence it will suffice to show that the conditions for existence of left adjoint are satisfied in this case.

Since $q : \mathcal{C}^\otimes \longrightarrow \mathcal{O}^\otimes$ is a left fibration q is **compatible with colimits indexed by weakly contractible diagrams** in the sense of [Lur2] Definition 3.1.1.18 (because weakly contractible colimits exist in every ∞ -groupoid and are preserved by any functor between ∞ -groupoids). Combining Corollary 3.1.3.4 and Proposition 3.1.1.20 of [Lur2] we see that the desired free algebra functor exists. \square

In view of Lemma 2.12 it will be enough to check that for every object $M \in \mathcal{O}\mathcal{F}$ (i.e. every connected 1-manifolds) the ∞ -category

$$\mathcal{F}_M \stackrel{\mathrm{def}}{=} \mathcal{O}\mathcal{J}_{\mathrm{act}}^\otimes \times_{\mathcal{O}\mathcal{F}_{\mathrm{act}}^\otimes} (\mathcal{O}\mathcal{F}_{\mathrm{act}}^\otimes)_{/M}$$

is weakly contractible.

Unwinding the definitions we see that the objects of \mathcal{F}_M are tuples of 1-manifolds (M_1, \dots, M_n) ($n \geq 1$), such that each M_i is diffeomorphic to a segment, together with an orientation preserving embedding

$$f : M_1 \coprod \dots \coprod M_n \hookrightarrow M$$

A morphism in \mathcal{F}_M from

$$f : M_1 \coprod \dots \coprod M_n \hookrightarrow M$$

to

$$g : M'_1 \coprod \dots \coprod M'_m \hookrightarrow M$$

is a π_0 -surjective orientation-preserving embedding

$$T : M_1 \coprod \dots \coprod M_n \longrightarrow M'_1 \coprod \dots \coprod M'_m$$

together with an **isotopy** $g \circ T \sim f$.

Now when M is the segment then \mathcal{F}_M contains a terminal object and so is weakly contractible. Hence we only need to take care of the case of the circle $M = S^1$.

It is not hard to verify that the category F_{S^1} is in fact discrete - the space of self isotopies of any embedding $f : M_1 \coprod \dots \coprod M_n \hookrightarrow M$ is equivalent to the loop space of S^1 and hence discrete. In fact one can even describe F_{S^1} in completely combinatorial terms. In order to do that we will need some terminology.

Definition 2.13. Let Λ_∞ be the category whose objects correspond to the natural numbers $1, 2, 3, \dots$ and the morphisms from n to m are (weak) order preserving maps $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $f(x + n) = f(x) + m$.

The category Λ_∞ is a model for the the universal fibration over the cyclic category, i.e., there is a left fibration $\Lambda_\infty \longrightarrow \Lambda$ (where Λ is connes' cyclic category) such that the fibers are connected groupoids with a single object having automorphism group \mathbb{Z} (or in other words circles). In particular the category Λ_∞ is known to be weakly contractible. See [Kal] for a detailed introduction and proof (Lemma 4.8).

Let $\Lambda_\infty^{\text{sur}}$ be the sub category of Λ_∞ which contains all the objects and only **surjective** maps between. It is not hard to verify explicitly that the map $\Lambda_\infty^{\text{sur}} \longrightarrow \Lambda_\infty$ is cofinal and so $\Lambda_\infty^{\text{sur}}$ is contractible as well. Now we claim that F_{S^1} is in fact equivalent to $\Lambda_\infty^{\text{sur}}$.

Let $\Lambda_{\text{big}}^{\text{sur}}$ be the category whose objects are linearly ordered sets S with an order preserving automorphisms $\sigma : S \longrightarrow S$ and whose morphisms are surjective order preserving maps which commute with the respective automorphisms. Then $\Lambda_\infty^{\text{sur}}$ can be considered as a full subcategory of $\Lambda_{\text{big}}^{\text{sur}}$ such that n corresponds to the object (\mathbb{Z}, σ_n) where $\sigma_n : \mathbb{Z} \longrightarrow \mathbb{Z}$ is the automorphism $x \mapsto x + n$.

Now let $p : \mathbb{R} \longrightarrow S^1$ be the universal covering. We construct a functor $F_{S^1} \longrightarrow \Lambda_{\text{big}}^{\text{sur}}$ as follows: given an object

$$f : M_1 \coprod \dots \coprod M_n \hookrightarrow S^1$$

of F_{S^1} consider the fiber product

$$P = \left[M_1 \coprod \dots \coprod M_n \right] \times_{S^1} \mathbb{R}$$

note that P is homeomorphic to an infinite union of segments and the projection

$$P \longrightarrow \mathbb{R}$$

is injective (because f is injective) giving us a well defined linear order on P . The automorphism $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ of \mathbb{R} over S^1 given by $x \mapsto x + 1$ gives an order preserving automorphism $\tilde{\sigma} : P \rightarrow P$.

Now suppose that $((M_1, \dots, M_n), f)$ and $((M'_1, \dots, M'_m), g)$ are two objects and we have a morphism between them, i.e. an embedding

$$T : M_1 \coprod \dots \coprod M_n \rightarrow M'_1 \coprod \dots \coprod M'_m$$

and an isotopy $\psi : g \circ T \sim f$. Then we see that the pair (T, ψ) determine a well defined order preserving map

$$[M_1 \coprod \dots \coprod M_n] \times_{S^1} \mathbb{R} \rightarrow [M'_1 \coprod \dots \coprod M'_m] \times_{S^1} \mathbb{R}$$

which commutes with the respective automorphisms. Clearly we obtain in this way a functor $u : F_{S^1} \rightarrow \Lambda_{\text{big}}^{\text{sur}}$ whose essential image is the same as the essential image of $\Lambda_{\infty}^{\text{sur}}$. It is also not hard to see that u is fully faithful. Hence F_{S^1} is equivalent to $\Lambda_{\infty}^{\text{sur}}$ which is weakly contractible. This finishes the proof of the theorem. \square

Let

$$\text{Alg}_{\mathcal{O}\mathcal{J}/\mathcal{B}_1^{\text{ev}}}^{\text{nd}}(\mathcal{F}_{\varphi}^{\otimes}) \subseteq \text{Alg}_{\mathcal{O}\mathcal{J}/\mathcal{B}_1^{\text{ev}}}(\mathcal{F}_{\varphi}^{\otimes})$$

denote the full sub ∞ -groupoid corresponding to the full sub ∞ -groupoid

$$\text{Alg}_{\mathcal{O}\mathcal{F}/\mathcal{B}_1^{\text{ev}}}^{\text{nd}}(\mathcal{F}_{\varphi}^{\otimes}) \subseteq \text{Alg}_{\mathcal{O}\mathcal{F}/\mathcal{B}_1^{\text{ev}}}(\mathcal{F}_{\varphi}^{\otimes})$$

under the equivalence of Theorem 2.11.

Now the last step of the cobordism hypothesis will be complete once we show the following:

Lemma 2.14. *The ∞ -groupoid*

$$\text{Alg}_{\mathcal{O}\mathcal{J}/\mathcal{B}_1^{\text{ev}}}^{\text{nd}}(\mathcal{F}_{\varphi}^{\otimes})$$

is contractible.

Proof. Let

$$q : p^* \mathcal{F}_{\varphi} \rightarrow \mathcal{O}\mathcal{J}^{\otimes}$$

be the pullback of left fibration $\mathcal{F}_{\varphi} \rightarrow \mathcal{B}_1^{\text{ev}}$ via the map $p : \mathcal{O}\mathcal{J}^{\otimes} \rightarrow \mathcal{B}_1^{\text{ev}}$, so that q is a left fibration as well. In particular, since $\mathcal{O}\mathcal{J}^{\otimes}$ is the non-unital associative ∞ -operad, we see that q classifies an ∞ -groupoid $q^{-1}(\mathcal{O}\mathcal{J})$ with a non-unital monoidal structure. Unwinding the definitions one sees that this ∞ -groupoid is the fundamental groupoid of the space

$$\text{Map}_{\mathbb{C}}(1, \varphi(X_+) \otimes \varphi(X_-))$$

where $X_+, X_- \in \mathcal{B}^{\text{ev}_1}$ are the points with positive and negative orientations respectively. The monoidal structure sends a pair of maps

$$f, f' : 1 \longrightarrow \varphi(X_+) \otimes \varphi(X_-)$$

to the composition

$$\begin{aligned} 1 &\xrightarrow{f \otimes f'} [\varphi(X_+) \otimes \varphi(X_-)] \otimes [\varphi(X_+) \otimes \varphi(X_-)] \xrightarrow{\simeq} \\ &\varphi(X_+) \otimes [\varphi(X_-) \otimes \varphi(X_+)] \otimes \varphi(X_-) \xrightarrow{Id \otimes \varphi(\text{ev}) \otimes Id} \varphi(X_+) \otimes \varphi(X_-) \end{aligned}$$

Since \mathcal{C} has duals we see that this monoidal ∞ -groupoid is equivalent to the fundamental ∞ -groupoid of the space

$$\text{Map}_{\mathcal{C}}(\varphi(X_+), \varphi(X_+))$$

with the monoidal product coming from **composition**.

Now

$$\text{Alg}_{\mathcal{OJ}/\mathcal{B}_1^{\text{ev}}}(\mathcal{F}_{\varphi}) \simeq \text{Alg}_{\mathcal{OJ}/\mathcal{OJ}}(p^*\mathcal{F}_{\varphi})$$

classifies \mathcal{OJ}^{\otimes} -algebra objects in $p^*\mathcal{F}_{\varphi}$, i.e. non-unital algebra objects in

$$\text{Map}_{\mathcal{C}}(\varphi(X_+), \varphi(X_+))$$

with respect to composition. The full sub ∞ -groupoid

$$\text{Alg}_{\mathcal{OJ}/\mathcal{B}_1^{\text{ev}}}^{\text{nd}}(\mathcal{F}_{\varphi}) \subseteq \text{Alg}_{\mathcal{OJ}/\mathcal{B}_1^{\text{ev}}}(\mathcal{F}_{\varphi})$$

will then classify non-unital algebra objects A which correspond to **self equivalences**

$$\varphi(X_+) \longrightarrow \varphi(X_+)$$

It is left to prove the following lemma:

Lemma 2.15. *Let \mathcal{C} be an ∞ -category. Let $X \in \mathcal{C}$ be an object and let \mathcal{E}_X denote the ∞ -groupoid of self equivalences $u : X \longrightarrow X$ with the monoidal product induced from composition. Then the ∞ -groupoid of non-unital algebra objects in \mathcal{E}_X is contractible.*

Proof. Let $\mathcal{A}ss_{\text{nu}}$ denote the non-unital associative ∞ -operad. The identity map $\mathcal{A}ss_{\text{nu}} \longrightarrow \mathcal{A}ss_{\text{nu}}$ which is in particular a left fibration of ∞ -operads classifies the terminal non-unital monoidal ∞ -groupoid \mathcal{A} which consists of single automorphismless idempotent object $a \in \mathcal{A}$. The non-unital algebra objects in \mathcal{E}_X are then classified by non-unital lax monoidal functors

$$\mathcal{A} \longrightarrow \mathcal{E}_X$$

Since \mathcal{E}_X is an ∞ -groupoid this is same as non-unital monoidal functors (without the lax)

$$\mathcal{A} \longrightarrow \mathcal{E}_X$$

Now the forgetful functor from unital to non-unital monoidal ∞ -groupoids has a left adjoint. Applying this left adjoint to \mathcal{A} we obtain the ∞ -groupoid \mathcal{UA} with two automorphismless objects

$$\mathcal{UA} = \{1, a\}$$

such that 1 is the unit of the monoidal structure and a is an idempotent object.

Hence we need to show that the ∞ -groupoids of monoidal functors

$$\mathcal{UA} \longrightarrow \mathcal{E}_X$$

is contractible. Now given a monoidal ∞ -groupoid \mathcal{G} we can form the ∞ -category $\mathcal{B}(\mathcal{G})$ having a single object with endomorphism space \mathcal{G} (the monoidal structure on \mathcal{G} will then give the composition structure). This construction determines a fully faithful functor from the ∞ -category of monoidal ∞ -groupoids and the ∞ -category of pointed ∞ -categories (see [Lur1] Remark 4.4.6 for a much more general statement). In particular it will be enough to show that the ∞ -groupoid of **pointed functors**

$$\mathcal{B}(\mathcal{UA}) \longrightarrow \mathcal{B}(\mathcal{E}_X)$$

is contractible. Since $\mathcal{B}(\mathcal{E}_X)$ is an ∞ -groupoid it will be enough to show that $\mathcal{B}(\mathcal{UA})$ is weakly contractible.

Now the nerve $N\mathcal{B}(\mathcal{UA})$ of $\mathcal{B}(\mathcal{UA})$ is the simplicial set in which for each n there exists a single **non-degenerate** n -simplex $\sigma_n \in N\mathcal{B}(\mathcal{UA})_n$ such that $d_i(\sigma_n) = \sigma_{n-1}$ for all $i = 0, \dots, n$. By Van-Kampen it follows that $N\mathcal{B}(\mathcal{UA})$ is simply connected and by direct computation all the homology groups vanish. \square

This finishes the proof of Lemma 2.14. \square

This finishes the proof of Theorem 2.6. \square

3 From Quasi-Unital to Unital Cobordism Hypothesis

In this section we will show how the quasi-unital cobordism hypothesis (Theorem 2.6) implies the last step in the proof of the 1-dimensional cobordism hypothesis (Theorem 1.8).

Let $M : \mathcal{B}_1^{\text{ev}} \longrightarrow \text{Grp}_\infty$ be a non-degenerate lax symmetric monoidal functor. We can construct a pointed **non-unital** symmetric monoidal ∞ -category \mathcal{C}_M as follows:

1. The objects of \mathcal{C}_M are the objects of $\mathcal{B}_1^{\text{ev}}$. The marked point is the object X_+ .
2. Given a pair of objects $X, Y \in \mathcal{C}_M$ we define

$$\text{Map}_{\mathcal{C}_M}(X, Y) = M(\check{X} \otimes Y)$$

Given a triple of objects $X, Y, Z \in \mathcal{C}_M$ the composition law

$$\mathrm{Map}_{\mathcal{C}_M}(\check{X}, Y) \times \mathrm{Map}_{\mathcal{C}_M}(\check{Y}, Z) \longrightarrow \mathrm{Map}_{\mathcal{C}_M}(\check{X}, Z)$$

is given by the composition

$$M(\check{X} \otimes Y) \times M(\check{Y} \otimes Z) \longrightarrow M(\check{X} \otimes Y \otimes \check{Y} \otimes Z) \longrightarrow M(\check{X} \otimes Z)$$

where the first map is given by the lax symmetric monoidal structure on the functor M and the second is induced by the evaluation map

$$\mathrm{ev}_Y : \check{Y} \otimes Y \longrightarrow 1$$

in $\mathcal{B}_1^{\mathrm{ev}}$.

3. The symmetric monoidal structure is defined in a straight forward way using the lax monoidal structure of M .

It is not hard to see that if M is non-degenerate then \mathcal{C}_M is **quasi-unital**, i.e. each object contains a morphism which **behaves** like an identity map (see [Har]). This construction determines a functor

$$G : \mathrm{Fun}_{\mathrm{nd}}^{\mathrm{lax}}(\mathcal{B}_1^{\mathrm{ev}}, \mathrm{Grp}_{\infty}) \longrightarrow \mathrm{Cat}_{\mathcal{B}_0^{\mathrm{un}}/}^{\mathrm{qu}, \otimes}$$

where $\mathrm{Cat}^{\mathrm{qu}, \otimes}$ is the ∞ -category of symmetric monoidal quasi-unital categories (i.e. commutative algebra objects in the ∞ -category $\mathrm{Cat}^{\mathrm{qu}}$ of quasi-unital ∞ -categories). In [Har] it is proved that the forgetful functor

$$S : \mathrm{Cat} \longrightarrow \mathrm{Cat}^{\mathrm{qu}}$$

From ∞ -categories to quasi-unital ∞ -categories is an **equivalence** and so the forgetful functor

$$S^{\otimes} : \mathrm{Cat}^{\otimes} \longrightarrow \mathrm{Cat}^{\mathrm{qu}, \otimes}$$

is an equivalence as well.

Now recall that

$$\mathrm{Cat}_{\mathcal{B}_1^{\mathrm{ev}}/}^{\mathrm{sur}} \subseteq \mathrm{Cat}_{\mathcal{B}_1^{\mathrm{ev}}/}^{\mathrm{nd}}$$

is the full subcategory spanned by essentially surjective functors $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{C}$. The fiber functor construction $\varphi \mapsto M_{\varphi}$ induces a functor

$$F : \mathrm{Cat}_{\mathcal{B}_1^{\mathrm{ev}}/}^{\mathrm{sur}} \longrightarrow \mathrm{Fun}_{\mathrm{nd}}^{\mathrm{lax}}(\mathcal{B}_1^{\mathrm{ev}}, \mathrm{Grp}_{\infty})$$

The composition $G \circ F$ gives a functor

$$\mathrm{Cat}_{\mathcal{B}_1^{\mathrm{ev}}/}^{\mathrm{sur}} \longrightarrow \mathrm{Cat}_{\mathcal{B}_0^{\mathrm{un}}/}^{\mathrm{qu}, \otimes}$$

We claim that $G \circ F$ is in fact **equivalent** to the composition

$$\mathrm{Cat}_{\mathcal{B}_1^{\mathrm{ev}}/}^{\mathrm{sur}} \xrightarrow{T} \mathrm{Cat}_{\mathcal{B}_0^{\mathrm{un}}/}^{\otimes} \xrightarrow{S} \mathrm{Cat}_{\mathcal{B}_0^{\mathrm{un}}/}^{\mathrm{qu}, \otimes}$$

where T is given by the restriction along $X_+ : \mathcal{B}_0^{\text{un}} \hookrightarrow \mathcal{B}_1^{\text{ev}}$ and S is the forgetful functor.

Explicitly, we will construct a natural transformation

$$N : G \circ F \xrightarrow{\cong} S \circ T$$

In order to construct N we need to construct for each non-degenerate functor $\varphi : \mathcal{B}_1^{\text{ev}} \rightarrow \mathcal{D}$ a natural pointed functor

$$N_\varphi : \mathcal{C}_{M_\varphi} \rightarrow \mathcal{D}$$

The functor N_φ will map the objects of \mathcal{C}_{M_φ} (which are the objects of $\mathcal{B}_1^{\text{ev}}$) to \mathcal{D} via φ . Then for each $X, Y \in \mathcal{B}_1^{\text{ev}}$ we can map the morphisms

$$\text{Map}_{\mathcal{C}_{M_\varphi}}(X, Y) = \text{Map}_{\mathcal{D}}(1, \check{X} \otimes Y) \rightarrow \text{Map}_{\mathcal{D}}(X, Y)$$

via the duality structure - to a morphism $f : 1 \rightarrow \check{X} \otimes Y$ one associates the morphism $\widehat{f} : X \rightarrow Y$ given as the composition

$$X \xrightarrow{Id \otimes f} X \otimes \check{X} \otimes Y \xrightarrow{\varphi(\text{ev}_X) \otimes Y} Y$$

Since \mathcal{D} has duals we get that N_φ is fully faithful and since we have restricted to essentially surjective φ we get that N_φ is essentially surjective. Hence N_φ is an equivalence of quasi-unital symmetric monoidal ∞ -categories and N is a natural equivalence of functors.

In particular we have a homotopy commutative diagram:

$$\begin{array}{ccc} & \text{Cat}_{\mathcal{B}_1^{\text{ev}}}^{\text{sur}} / & \\ F \swarrow & & \searrow T \\ \text{Fun}_{\text{nd}}^{\text{lax}}(\mathcal{B}_1^{\text{ev}}, \text{Grp}_\infty) & & \text{Cat}_{\mathcal{B}_0^{\text{un}}}^{\otimes} / \\ G \searrow & & \swarrow S \\ & \text{Cat}_{\mathcal{B}_0^{\text{un}}}^{\text{qu}, \otimes} / & \end{array}$$

Now from Lemma 1.6 we see that T is fully faithful. Since S is an equivalence of ∞ -categories we get

Corollary 3.1. *The functor $G \circ F$ is fully faithful.*

We are now ready to complete the proof of 1.8. Let \mathcal{D} be a symmetric monoidal ∞ -category with duals and let $\varphi : \mathcal{B} \rightarrow \mathcal{D}$ be a non-degenerate functor. We wish to show that the space of maps

$$\text{Map}_{\text{Cat}_{\mathcal{B}_1^{\text{ev}}}^{\text{sur}}}(\iota, \varphi)$$

is contractible. Consider the sequence

$$\mathrm{Map}_{\mathrm{Cat}_{\mathcal{B}_1^{\mathrm{ev}}/\mathrm{ev}}}^{\mathrm{sur}}(\iota, \varphi) \longrightarrow \mathrm{Map}_{\mathrm{Fun}_{\mathrm{nd}}^{\mathrm{lax}}(\mathcal{B}_1^{\mathrm{ev}}, \mathrm{Grp}_{\infty})}(M_{\iota}, M_{\varphi}) \longrightarrow \mathrm{Map}_{\mathrm{Cat}_{\mathcal{B}_0^{\mathrm{un}}/\mathrm{un}}}^{\mathrm{qu}, \otimes}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D})$$

By Theorem 2.6 the middle space is contractible and by lemma 3.1 the composition

$$\mathrm{Map}_{\mathrm{Cat}_{\mathcal{B}_1^{\mathrm{ev}}/\mathrm{ev}}}^{\mathrm{sur}}(\iota, \varphi) \longrightarrow \mathrm{Map}_{\mathrm{Cat}_{\mathcal{B}_0^{\mathrm{un}}/\mathrm{un}}}^{\mathrm{qu}, \otimes}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D})$$

is a weak equivalence. Hence we get that

$$\mathrm{Map}_{\mathrm{Cat}_{\mathcal{B}_1^{\mathrm{ev}}/\mathrm{ev}}}^{\mathrm{sur}}(\iota, \varphi)$$

is contractible. This completes the proof of Theorem 1.8.

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